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Higher bundle gerbes and cohomology classes in gauge theories

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Abstract

The notion of a higher bundle gerbe is introduced to give a geometric realisation of the higher degree integral cohomology of certain manifolds. We consider examples using the infinite-dimensional spaces arising in gauge theories.

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1. Introduction

This paper develops ideas hinted at in [4,12]. In order to make this account self contained we will review in Sections 2 and 3 relevant aspects of these earlier papers. We are interested in the general problem of realising higher degree cohomology classes of manifolds geometrically. The work of Brylinski [2] provides one approach to this problem via a sheaf theoretic description of the category theorists notion of a gerbe. In [12] a simpler approach, which seems sufficient for the applications we have in mind, was introduced by one of us. This simpler notion of 'bundle gerbe' enables us to realise classes in $H^3(M, Z)$. The question of what to do with higher degree classes was posed in [12] and it was conjectured that a notion of bundle n-gerbe was needed with 1-gerbes corresponding to the case described in [4]. (Line bundles should be regarded as 0-gerbes in this setting.)

The main result of the present paper is to show that there is indeed a notion of bundle n-gerbe (strictly speaking we discuss in detail the general definition for 2-gerbes only). Our

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motivation comes from the interesting examples [15] of de Rham forms on the space of connections obtained from Chern–Simons secondary characteristic classes exploited in the physics literature to study the cohomology of gauge groups. We do not completely resolve the connection between bundle gerbes and the Chern–Simons classes here nevertheless we provide a convincing application of the bundle 2-gerbe notion using examples motivated by gauge theories.

The main results of this paper can be summarised as follows. In Section 2 we extend the discussion of [12] in two ways. First we give an explicit proof of associativity of the product for bundle 1-gerbes in a form which can easily be generalised to the 2-gerbe situation. We also construct the tautological bundle 1-gerbe on manifolds which are not 2-connected. In Section 3 we develop the definition of a bundle 2-gerbe. This is related to the structures introduced in Freed [9] in his study of gauge theories. The discussion in Section 2 generalises in a straightforward way to the case of bundle 2-gerbes. In Section 4 we review [4] and then provide some examples, in the context of gauge field theories, of the gerbe viewpoint. The most novel example is the case of degree 4 de Rham cohomology of the space of connections modulo the gauge action or equivalently via transgression of a 3-cocycle on the gauge group. This is given a geometric realisation in this paper.

2. Bundle 1-gerbes

In [12] bundle gerbes were developed to provide an alternative geometric realisation of three-dimensional cohomology to that given by Brylinski's sheaf theoretic approach to gerbes [2]. In this paper we generalise this definition to that of a bundle n-gerbe which maybe used to give a geometric realisation of cohomology in degree $n+2$. First we review the construction in [4]. At various points we refer to $H^p(M, Z)$ when what we really mean is the image in the de Rham cohomology of the integral cohomology of M. We review first the theory of bundle gerbes.

2.1. Review of bundle 1-gerbes

Consider a fibration $\pi : Y \to M$. Denote by Y_m the fibre of Y over m, that is the set $\pi^{-1}(m)$. The fibre product of Y with itself, denoted $Y^{[2]} = Y \times_\pi Y$, is a new fibration whose fibre at m is $(Y^{[2]})_m = Y_m \times Y_m$. It is often useful to think of it as the subset of pairs (y_1, y_2) in $Y \times Y$ such that $\pi(y_1) = \pi(y_2)$. A bundle gerbe is a principal $C^\times$-bundle P over $Y^{[2]}$ with a composition map defined fibre by fibre smoothly as

$$P_{(x,y)} \times P_{(y,z)} \longrightarrow P_{(x,z)}. \tag{2.1}$$

This composition map (2.1) is a morphism of the bundle $P \otimes P$ over the groupoid $Y^{[2]} \circ Y^{[2]} \longrightarrow Y^{[2]}$ which satisfies associativity, where $Y^{[2]} \circ Y^{[2]}$ is the set of pairs $((y_1, y_2), (y_2, y_3))$. It is shown in [12] that a bundle gerbe also has an identity and an inverse. The identity is a section of P over the diagonal inside $Y^{[2]}$ and the inverse is a bundle map $P_{(x,y)} \to P_{(y,x)}$ denoted by $p \mapsto p^{-1}$ such that $(pz)^{-1} = p^{-1}z^{-1}$.

We denote a bundle gerbe on M by the diagram

$$\begin{array}{ccc} P & & \\ \downarrow & & \\ Y_1^{[2]} & \overset{\pi_i}{\rightrightarrows} & Y_1 \\ & & \downarrow \\ & & M \end{array} \tag{2.2}$$

where π_i $(i = 1, 2)$ denote the projections onto the various factors.

A bundle gerbe connection is a connection on $P \to Y^{[2]}$ which respects the gerbe structure, this means that over the diagonal it is flat, that the product map on the bundle gerbe sends the connection to itself and that the inverse map sends the connection to its dual. To understand how to extract the Dixmier–Douady class of the bundle gerbe from this connection we need to digress for a moment. Given a fibration $Y \to M$ we can form repeated fibre products $Y^{[p]}$ whose fibre at m is Y_m^p. We call this the pth fibre product. There are p projections $\pi_i : Y^{[p]} \to Y^{[p-1]}$, each of which just omits a factor. Pulling back differential q forms with these and adding with an alternating sign defines a map $\delta : \Omega^q(Y^{[p]}) \to \Omega^q(Y^{[p-1]})$. This is in fact a chain map (i.e. $\delta^2 = 0$) and defines a complex:

$$\Omega^q(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \cdots \tag{2.3}$$

It was shown in [12] that this complex is exact. The requirement that the bundle gerbe connection be compatible with the product implies that its curvature $F \in \Omega^2(Y^{[2]})$ satisfies $\delta(F) = 0$ and hence $F = \delta(f)$ or $F = \pi_1^* f - \pi_2^* f$ for some two-form on Y. Then we have $\delta(\mathrm{d}f) = \mathrm{d}\delta(f) = \mathrm{d}F = 0$ so that $\mathrm{d}f = \pi^*(\omega)$ where ω is a closed three-form on M. The three-form $(1/2\pi i)\omega$ defines an element in $H^3(M, Z)$ which is the Dixmier–Douady class of the gerbe [12]. If $(1/2\pi i)\omega$ is cohomologous to zero in $H^3(M, Z)$, then

$$P \cong \pi_1^*(L) \otimes \pi_2^*(L^{-1}), \tag{2.4}$$

where L is some $C^\times$-bundle over Y. We call such a bundle gerbe, a trivial bundle gerbe. For the details, see [12].

A more cohomological treatment of this construction can be obtained by considering the short sequence

$$Y^{[2]} \overset{\pi_i}{\rightrightarrows} Y^{[1]} \longrightarrow M, \tag{2.5}$$

which we can use to define a short exact sequence of de Rham complexes:

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\pi^*} \Omega^*(Y^{[1]}) \xrightarrow{\pi_1^* - \pi_2^*} \Omega^*(Y^{[2]}) \cap Im(\pi_1^* - \pi_2^*) \longrightarrow 0. \tag{2.6}$$

This induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^q(M) \xrightarrow{\pi^*} H^q(Y^{[1]}) \xrightarrow{\pi_1^* - \pi_2^*} H_\pi^q(Y^{[2]}) \xrightarrow{\triangle} H^{q+1}(M) \longrightarrow \cdots, \tag{2.7}$$

where $H^q_\pi(Y^{[2]}) = \{\omega \in H^q(Y^{[2]}) \mid \omega = (\pi_1^* - \pi_2^*) f \text{ for } f \in \Omega^q(Y^{[1]})\}$. In fact, from the discussion above we see that the Chern class of the bundle $Q \to Y_2^{[2]}$ is in $H^2_\pi(Y^{[2]}, Z) \subset H^2(Y^{[2]}, Z)$ and the construction of the Dixmier–Douady class we described is just the application of the map

$$H^2_\pi(Y^{[2]}, Z) \xrightarrow{\Delta} H^3(M, Z). \tag{2.8}$$

This is analogous to the Chern–map for line bundles.

When the manifold M is 2-connected there is an explicit realisation of a bundle gerbe associated with $(1/2\pi i)\omega \in H^3(M, Z)$, called the tautological bundle gerbe [12]. We review that construction here in a slightly different form so that we can prove explicitly the associativity not proved in [12]. Fix a basepoint x_0 in M and consider the based path fibration:

$$Y = \mathcal{P}_0 M = \{\rho : [0, 1] \to M \mid \rho(0) = x_0\}, \tag{2.9}$$

$\pi : Y \to M$ is given by $\pi(\rho) = \rho(1)$. Then $Y^{[2]}$ is the space of pairs of smooth paths starting from x_0 and ending with the same end point. We can construct a $C^\times$ bundle Q over $Y^{[2]}$ by defining the fibre at (ρ_0, ρ_1) to be the space whose elements are equivalence classes of pairs $[\mu, z]$ where $\mu : [0, 1] \times [0, 1] \to X$ is a piecewise smooth homotopy, with endpoints fixed, from ρ_0 to ρ_1 and z is a non-zero complex number. Recall that a homotopy with endpoints fixed satisfies $\mu(s, 0) = \rho_0(0) = \rho_1(0) = x_0$, $\mu(s, 1) = \rho_0(1) = \rho_1(1)$ for all s and $\mu(0, t) = \rho_0(t)$ and $\mu(1, t) = \rho_1(t)$ for all t. We say two pairs (μ, z) and (μ', z') are equivalent if for any homotopy

$$F : [0, 1] \times [0, 1] \times [0, 1] \to M$$

between μ and μ' we have $\exp(\int F^*(\omega))z = z'$, where the integral is over $[0, 1] \times [0, 1] \times [0, 1]$. The condition on the homotopy F is that $F(0, s, t) = \mu(s, t)$, $F(1, s, t) = \mu'(s, t)$ and for each r we have that $F(r, \ , \ )$ is a homotopy with endpoints fixed between ρ_0 and ρ_1.

We want to now construct the product

$$Q_{(\rho_1,\rho_2)} \otimes Q_{(\rho_2,\rho_3)} \to Q_{(\rho_1,\rho_3)}.$$

If α is a homotopy from ρ_1 to ρ_2 and β is a homotopy from ρ_2 to ρ_3 then we can construct a homotopy $\alpha \circ \beta$ from ρ_1 to ρ_2 in the usual way by letting $\alpha \circ \beta(s, t)$ equals $\alpha(2s, t)$ for s between 0 and $\frac{1}{2}$ and all t and letting $\alpha \circ \beta(s, t)$ equals $\beta(2s - 1, t)$ for s between $\frac{1}{2}$ and 1 and all t. We need to check that this map is well-defined, that is it respects the equivalence relation defining Q. Consider then α' and β' with F a homotopy from α to α' and G a homotopy from β to β'. We can construct a homotopy $F * G$ from $\alpha \circ \beta$ to $\alpha' \circ \beta'$ by letting

$$(F \circ G)(r, \ , \ ) = F(r, \ , \ ) \circ G(r, \ , \ )$$

for all r. Since the linear change defining $\circ$ is a diffeomorphism we deduce that

$$\int (F \circ G)^*(\omega) = \int (F)^*(\omega) + \int (G)^*(\omega)$$

and it follows that if $[\alpha, z] = [\alpha', z']$ and $[\beta, z] = [\beta', w']$ then $[\alpha * \beta, zw] = [\alpha' * \beta', z'w']$. The product on Q is then defined by

$$[\alpha, z] \otimes [\beta, z] \mapsto [\alpha \circ \beta, zw].$$

We want to prove that this product is associative. Let ρ_4 be another path and let γ be a homotopy between ρ_3 and ρ_4. It suffices to prove that $(\alpha \circ \beta) \circ \gamma$ and $\alpha \circ (\beta \circ \gamma)$ are homotopic to each other by a homotopy F satisfying $F^*(\omega) = 0$. To construct the homotopy let r be a point in $[0, 1]$ and consider the division of $[0, 1]$ into three intervals

$$[0, (1+r)\tfrac{1}{4}], \quad [(1+r)\tfrac{1}{4}, (1+r)\tfrac{1}{2}], \quad [(1+r)\tfrac{1}{2} 1]$$

and let $m_r(\alpha, \beta, \gamma)(\,, t)$ be the homotopy obtained by applying $\alpha(\,, t)$ $\beta(\,, t)$ and $\gamma(\,, t)$ at an appropriately scaled speed to each of these intervals, respectively. So, in particular, $m_0(\alpha, \beta, \gamma) = (\alpha \circ \beta) \circ \gamma$ and $m_1(\alpha, \beta, \gamma) = \alpha \circ (\beta \circ \gamma)$. Then define

$$F(r, s, t) = m_r(\alpha, \beta, \gamma)(s, t).$$

Notice that the image of F is, at best, two-dimensional so that the pull back of the three-form ω is zero, as required. In fact it is not difficult to define the lines in $[0, 1] \times [0, 1] \times [0, 1]$ along which F is constant. It follows that the product is associative as is required to define a bundle gerbe.

The well-known example of a bundle gerbe is given by the central extension of the loop group [13], which is a realisation of a degree 3 cohomology element, namely the generator of $H^3(G, Z)$:

$$(\xi_1, \xi_2, \xi_3) \longmapsto \frac{1}{8\pi^2} \langle \xi_1, [\xi_2, \xi_3] \rangle \tag{2.10}$$

2.2. The tautological construction: non-connected case

When the manifold M is not 2-connected, we cannot apply the above procedure to construct the tautological bundle gerbe. But if we assume that $\pi_2(M)$ has no non-trivial $C^\times$-extensions, we can still realise the tautological bundle gerbe construction in a slightly different way.

Now it is not difficult to see that we may identify the space $Y^{[2]}$ of $\mathcal{P}_0(M)$ over M as the loop space $S^1(M)$ of piecewise smooth loops in M. We may assume that M is simply connected (for if it is not, then we simply perform the construction below over each connected component of $S^1(M)$). Through the evaluation map, we can pull back the three-form ω on M to $S^1(M) \times [0, 1]$ and integrate out $[0, 1]$ to obtain an integral two-form Θ on $S^1(M)$. Note that $\pi_1(S^1(M)) = \pi_2(M)$. Denote the universal covering space as $\widetilde{S}^1(M)$. Pull back Θ to $\widetilde{S}^1(M)$, and denote the pulled back form by $\widetilde{\Theta}$, which is $\pi_2(M)$-invariant. Since $\widetilde{S}^1(M)$ is simply connected, we can construct a $C^\times$-bundle $\widetilde{Q}$ over $\widetilde{S}^1(M)$ by the tautological method. Now it is fairly standard in the theory of line bundles (see however [7] for a discussion) that the fundamental group of $\widetilde{S}^1(M)$ (here $\pi_2(M)$) has a $C^\times$-extension $\widetilde{\pi}_2(M)$, which acts on $\widetilde{Q}$. By assumption, $\pi_2(M)$ has only trivial $C^\times$-extensions, therefore we can view $\pi_2(M)$ as a subgroup in $\widetilde{\pi}_2(M)$, quotient $\widetilde{Q}$ by this subgroup, we get a $C^\times$-bundle Q over $S^1(M)$.

To verify that the quotient bundle is a bundle gerbe, we use the expression for the lifted action of $\pi_2(M)$ [7]. Recall that the universal cover of $S^1(M)$, $\widetilde{S}^1(M)$, consists of the based homotopic discs in M, that is, over the point $\gamma \in S^1(M)$, the fibre is the homotopic disc bounded by γ. There is a natural commutative diagram which gives the groupoid structure in $\widetilde{S}^1(M)$:

$$\begin{array}{ccccc} \widetilde{S}^1(M) & \circ & \widetilde{S}^1(M) & \longrightarrow & \widetilde{S}^1(M) \\ & & \downarrow & & \downarrow \\ S^1(M) & \circ & S^1(M) & \longrightarrow & S^1(M) \end{array}$$

The line bundle $\widetilde{Q}$ is given by triples $([D_1], p_{[D_1]}, z_1)$ modulo the equivalence relation given by the two-form $\widetilde{\Theta}$, where $p_{[D_1]}$ is the path from the base point $[D_0]$ in $\widetilde{S}^1(M)$ to $[D_1]$. For $\psi \in \pi_1(S^1(M)) = \pi_2(M)$, the lifted action (under the assumption that $\pi_2(M)$ has no non-trivial $C^\times$-extension) $\hat{\psi}$ is calculated in [7] as follows:

$$\hat{\psi} \cdot ([D_1], p_{[D_1]}, z_1) = (\psi([D_1]), p_\psi * \psi(p_{[D_1]}), z_1),$$

where p_ψ is the fixed path (only depending on ψ) from $[D_0]$ to $\psi([D_0])$ which is defined by the splitting map for the exact sequence:

$$0 \to C^\times \to \widetilde{\pi}_2(M) \to \pi_2(M) \to 0.$$

The product structure on $\widetilde{Q}$ is given by

$$([D_1], p_{[D_1]}, z_1) \circ ([D_2], p_{[D_2]}, z_2) = ([D_1 \circ D_2], p_{[D_1]} \circ p_{[D_2]}, z_1 z_2).$$

We only need to prove that the above product structure on $\widetilde{Q}$ is a $\pi_2(M)$-homomorphism. This is just the following identity for the groupoid structure on $\widetilde{S}^1(M)$:

$$p_\psi * \psi(p_{[D_1]} \circ p_{[D_2]}) = p_\psi * \psi(p_{[D_1]}) \circ p_\psi * \psi(p_{[D_2]}).$$

Therefore the gerbe structure on $\widetilde{S}^1(M)$ descends to a gerbe structure on $S^1(M)$.

3. Bundle 2-gerbes

To generalise these ideas to higher gerbes we note that $Y^{[2]}$ is just the space $S^1(M)$ of based loops in M. The two projection maps project a loop to the two paths corresponding to restricting to the upper and lower 'hemispheres' of S^1. The usual evaluation map

$$S^1(M) \times [0, 1] \to M \tag{3.1}$$

allows us to pull back ω and integrate out the $[0, 1]$ to obtain a two-form on $S^1(M)$. This is the curvature considered above.

Hence (2.2) becomes the diagram:

$$\begin{array}{ccc} Q & & \\ \downarrow & & \\ S^1(M) & \overset{\pi_i}{\rightrightarrows} & D^1(M) \\ & & \downarrow \\ & & M \end{array} \tag{3.2}$$

An obvious generalisation of this which works for 3-connected manifolds M is to consider the diagram:

$$\begin{array}{ccccc} Q & & & & \\ \downarrow & & & & \\ S^2(M) & \overset{\pi_i}{\rightrightarrows} & D^2(M) & & \\ & & \downarrow & & \\ & & S^1(M) & \overset{\pi_i}{\rightrightarrows} & D^1(M) \\ & & & & \downarrow \\ & & & & M \end{array} \tag{3.3}$$

Here $S^n(M)$ is the space of based maps from n spheres to M with the base point on the equator and $D^n(M)$ the space of based maps from the n-dimensional ball to M with the base point on the boundary. By restricting to the upper and lower hemispheres of S^n we obtain a pair of projections

$$S^n(M) \overset{\pi_i}{\rightrightarrows} D^n(M). \tag{3.4}$$

Restricting to the boundary of the n disc also defines a map

$$\pi: D^n(M) \to S^{n-1}(M). \tag{3.5}$$

This means that $D^n(M)$ is a fibre bundle over $S^{n-1}(M)$ whose fibre $D^n_f(M)$ at $f : S^{n-1} \to M$ is all the extensions of f to the $n+1$-dimension ball and we have that

$$S^n(M) = D^n(M) \times_\pi D^n(M). \tag{3.6}$$

The induced projection $S^n(M) \to S^{n-1}(M)$ is that induced by restriction to the equator. The line bundle Q is defined in a manner analogous to the bundle 1-gerbe case or equivalently we use the evaluation map

$$S^2(M) \times S^2 \to M. \tag{3.7}$$

Pull back with this and integrate over the S^2 to define a two-form F on $S^2(M)$. This is closed and integral and hence defines a line bundle which is Q. We call (3.3) the tautological bundle 2-gerbe.

Now, from the viewpoint of de Rahm cohomology the construction above works as follows. First we construct a sequence of forms by starting with $\Theta \in H^4(M, Z)$, then set

$$F = \int_{S^2} ev^*(\Theta) \in H^2(S^2(M), Z), \qquad f_1 = \int_{D^2} ev^*(\Theta) \in \Omega^2(D^2(M)),$$
$$\omega = \int_{S^1} ev^*(\Theta) \in H^3(S^1(M), Z), \qquad f_2 = \int_{D^1} ev^*(\Theta) \in \Omega^3(D^1(M)).$$

It is easy to show that the above forms satisfy:

$$\begin{aligned} F &= (\pi_1^* - \pi_2^*) f_1, \qquad \mathrm{d} f_1 = \pi^*(\omega), \\ \omega &= (\pi_1^* - \pi_2^*) f_2, \qquad \mathrm{d} f_2 = \pi^*(\Theta). \end{aligned} \tag{3.8}$$

Recalling the definition of Q above, $Q = D^3(M) \times C^\times / \sim$ where the equivalence relation is given by

$$(B_1, z_1) \sim (B_2, z_2)$$

if and only if $\partial B_1 = \partial B_2 \in S^2(M)$ and $z_1 = z_2 \exp(\int_{D^4} \Theta)$, with D^4 being the four-dimensional disc bounded by the 3-sphere formed by gluing B_1 and B_2 along the common boundary. One may easily check that the definition is independent of the choice of D^4. It is now straightforward, using the methods of Section 2, to define a product on Q and show that Q is a bundle gerbe over $S^1(M)$ whose Dixmier–Douady class is $\omega = \int_{S^1} ev^*(\Theta)$ in $H^3(S^1(M), Z)$ (from the first two identities in (3.8)). Indeed it suffices really to note that $D^2(M)$ is the space of based paths in $S^1(M)$ and that the rest of the construction is just the tautological bundle gerbe for the case of the three-form $\omega = \int_{S^1} ev^*(\Theta)$.

In the present situation as well as the bundle gerbe structure there is additional structure in the form of a multiplication on the fibration $D^2(M) \to S^1(M)$ which can be lifted to Q. Specifically, for two loops (γ_1, γ_2) and (γ_2, γ_3) in $S^1(M)$, one can define

$$(\gamma_1, \gamma_2) * (\gamma_2, \gamma_3) = (\gamma_1, \gamma_3).$$

This product can be lifted to the total space of the fibration $D^2(M) \to S^1(M)$ as follows. If we think of two bounding discs in $D^2(M)$ as homotopies μ and ρ from γ_1 to γ_2 and γ_2 to γ_3 then we can compose them as in Section 1 to get a homotopy $\mu * \rho$ from γ_1 to γ_3. Hence we have a product:

$$m : D^2(M)_{\gamma_1} \times D^2(M)_{\gamma_2} \longrightarrow D^2(M)_{\gamma_1 \circ \gamma_2}.$$

Note that this product is not associative. The product m also defines a product

$$m : S^2(M)_{\gamma_1} \times S^2(M)_{\gamma_2} \longrightarrow S^2(M)_{\gamma_1 \circ \gamma_2}.$$

This product can be lifted to a product on Q as follows. Let $[B_1, z_1] \in Q_{(\mu_1, \rho_1)}$ and $[B_2, z_2] \in Q_{(\mu_2 \rho_2)}$. Then we can think of B_i as a homotopy from μ_i to ρ_i and we can define a homotopy from $\mu_1 * \rho_1$ to $\mu_2 * \rho_2$ by $(B_1 * B_2)(s, \ ) = B_1(s, \ ) * B(s, \ )$. The product is then

$$[B_1, z_1] * [B_2, z_2] = [B_1 * B_2, z_1 z_2].$$

It is straightforward (but very tedious) using the methods of Section 2 to check that this product is well-defined and associative. In a similar fashion we can show that this product covers the product m above. So, we have lifted the product on $S^1(M)$ to a product on the bundle gerbe over $S^1(M)$. It is therefore natural to introduce the following:

Definition 3.1. A bundle 2-gerbe is a diagram of spaces of the form

$$\begin{array}{ccccc} Q & & & & \\ \downarrow & & & & \\ Y_2^{[2]} & \overset{\pi_i}{\rightrightarrows} & Y_2 & & \\ & & \downarrow & & \\ & & Y_1^{[2]} & \overset{\pi_i}{\rightrightarrows} & Y_1 \\ & & & & \downarrow \\ & & & & M \end{array} \tag{3.9}$$

where

$$\begin{array}{ccc} Q & & \\ \downarrow & & \\ Y_2^{[2]} & \overset{\pi_i}{\rightrightarrows} & Y_2 \\ & & \downarrow \\ & & Y_1^{[2]} \end{array} \tag{3.10}$$

is required to be a bundle 1-gerbe and the natural product on $Y_1^{[2]}$ is covered by the product on Q. That is, there is a fibration composition map over the groupoid $Y_1^{[2]} \circ Y_1^{[2]} \longrightarrow Y_1^{[2]}$ defined fibre by fibre as

$$m : (Y_2)_{(y_1, y_2)} \times (Y_2)_{(y_2, y_3)} \longrightarrow (Y_2)_{(y_1, y_3)}, \tag{3.11}$$

where $(y_1, y_2), (y_2, y_3) \in Y_1^{[2]}$ and this map m is compatible with the multiplication in Q.

It is clear now that we could extend the tower of spaces in (3.10) and define bundle n-gerbes however we leave this refinement to the reader (note that this is not a trivial extension as one needs to keep track of the product at each level). Finally we note that we may handle the non-connected tautological bundle 2-gerbe construction in a fashion similar to that for the bundle 1-gerbe. The space $D^2(M)$ is defined in the obvious fashion over each connected component of $S^1(M)$. One defines a gerbe structure using the WZW construction on the simply connected covering space of (each component) of $S^2(M)$ and under the assumption that the fundamental group of $S^2(M)$ has no non-trivial $C^\times$-extensions one may factor out by an action of this fundamental group to obtain the tautological bundle 2-gerbe.

4. Transgression

In [4] two of us introduced a twist on the usual transgression map arising from the contractibilty of the space of connections on a principal bundle. In our approach we use transgression to define a map from the pth cohomology of connection space modulo gauge transformations to the $(p-1)$th cohomology of the Lie algebra of the gauge group with coefficients in functions on the space of connections. This unifies the two views of anomalies as manifestations of the non-trivial topology of the space of connections modulo gauge transformations on the one hand and group cohomology of the gauge group on the other.

Let (P, G, M) be the G-principal bundle over a compact Riemannian manifold with a compact structure group G, let $\mathcal{A}$ denote the affine space of connections, modelled on AdP-valued 1-forms on M, $\Omega^1_M(AdP)$, where the AdP is the associated bundle by the adjoint representation of G on its Lie algebra $L(G)$. Denote by $\mathcal{G}$ the gauge group, viewed as the automorphisms of P, respecting the fibres and covering the identity map of M, $L(\mathcal{G})$ its Lie algebra which can be identified with $\Omega^0_M(AdP)$ the AdP-valued functions on M. By suitable basepointing, $\mathcal{G}$ acts on $\mathcal{A}$ freely.

In [4], we introduced the transgression map:

$$H^p(\mathcal{A}/\mathcal{G}, R) \longrightarrow H^{p-1}(L(\mathcal{G}), Map(\mathcal{A}, R)), \tag{4.1}$$

where $H^p(\mathcal{A}/\mathcal{G}, R)$ is the pth de Rham cohomology group of the moduli space $\mathcal{A}/\mathcal{G}$, and the notation

$$H^{p-1}(L(\mathcal{G}), Map(\mathcal{A}, R))$$

means the $(p-1)$th cohomology group of the Lie algebra $L(\mathcal{G})$ with values in $Map(\mathcal{A}, R)$. This transgression map is defined as follows. If $\omega \in \Omega^p(\mathcal{A}/\mathcal{G})$ with $\mathrm{d}\omega = 0$, then the pull back $\pi^*\omega$ in $\Omega^p(\mathcal{A})$ is an exact form on $\mathcal{A}$ due to the fact that $\mathcal{A}$ is an affine space. So $\pi^*\omega = \mathrm{d}\mu$ where δ is the exterior differential operator on $\mathcal{A}$ and $\mu \in \Omega^{p-1}(\mathcal{A})$. Consider $\mathcal{A} = \bigcup_A \mathcal{G} \cdot A$, then the vector field induced by the infinitesimal gauge transformation ϵ is $-\mathrm{d}_A\epsilon$ and the $p-1$-cocycle corresponding to ω is given by

$$(\epsilon_1, \dots, \epsilon_{p-1}) \longmapsto (-1)^{p-1}\mu(\mathrm{d}_A\epsilon_1, \dots, \mathrm{d}_A\epsilon_{p-1}). \tag{4.2}$$

The Lie algebra coboundary operator corresponds to the exterior differential operator on the de Rham complex and so (4.2) induces the transgression map (4.1).

4.1. The Atiyah–Singer construction of closed forms on $\mathcal{A}/\mathcal{G}$

There is a universal bundle $\mathcal{L}$ over $M \times (\mathcal{A}/\mathcal{G})$ [1,8] described as follows. At $(p, A) \in M \times \mathcal{A}$, the connection A gives a decomposition of the tangent space $T_p(P) = T_{[p]}(M) \oplus Lie(G)$ which gives a G-Riemannian metric on P, this metric and the correspoding Hodge $*$ operator endows $M \times \mathcal{A}$ with a Riemannian metric which is invariant under $\mathcal{G} \times G$. Now construct $\mathcal{L} = (M \times \mathcal{A})/\mathcal{G}$ then the universal bundle is $\mathcal{L}$ with base space $\mathcal{L}/G$ with its natural connection θ. The curvature $\mathcal{F}$ of θ has several components according to the degree in the M and $\mathcal{A}/\mathcal{G}$ directions.

Recall that the tangent space of $\mathcal{A}/\mathcal{G}$ at $[A]$ consists of those $Lie(G)$-valued one-forms on P which lie in the kernel of D_A^*. Next we note that

$$\mathcal{F} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2},$$

where

(1) $\mathcal{F}^{2,0}_{(x,A)} = F_A$,

(2) $\mathcal{F}^{1,1}_{(x,A)}(X,\xi) = \xi(X)$ for $X \in T_x(M), \xi \in T_{[A]}(\mathcal{A}/\mathcal{G})$,

(3) $\mathcal{F}^{0,2}_{(x,A)}(\xi,\eta) = -(D_A^* D_A)^{-1}(*(\xi \wedge *\eta))$ where $\xi,\eta \in T_{[A]}(\mathcal{A}/\mathcal{G})$.

For dim $M = 3$, we can construct a closed three-form on $\mathcal{A}/\mathcal{G}$ by considering

$$\int_M tr(\mathcal{F})^3. \tag{4.3}$$

The only contribution is from the terms

$$2\, str(\mathcal{F}^{2,0} \wedge \mathcal{F}^{1,1} \wedge \mathcal{F}^{0,2}) + tr(\mathcal{F}^{1,1})^3.$$

For the case dim $M = 4$ we again construct a closed four-form on $\mathcal{A}/\mathcal{G}$ by integration

$$\int_M tr(\mathcal{F})^4 \tag{4.4}$$

and the only contributing terms are

$$str(F^{0,2} \wedge F^{0,2} \wedge \mathcal{F}^{0,2} \wedge \mathcal{F}^{0,2}) + tr(\mathcal{F}^{1,1})^4 + str(F^{2,0} \wedge \mathcal{F}^{1,1} \wedge \mathcal{F}^{1,1} \wedge \mathcal{F}^{0,2}).$$

(Here str means the symmetric trace). Suitably normalised (4.3) and (4.4) define forms $\Theta_n, n = 3,4$, which determine integral cohomology classes on $\mathcal{A}/\mathcal{G}$. Then, using the viewpoint of higher bundle gerbes, Θ_n defines the tautological bundle $n-2$-gerbe on $\mathcal{A}/\mathcal{G}$. We discuss the $n = 3$ cases in more detail in Section 4.2 and the $n = 4$ case in Section 5.

4.2. A 1-gerbe on $\mathcal{A}/\mathcal{G}$ and the Faddeev–Mickelsson cocycle

In this section, we give some details of the tautological bundle 1-gerbe derived from the form (4.3) and give various geometric interpretations including relating it to the Faddeev–Mickelsson cocycle which gives rise to an extension of the gauge group.

Suppose M is a three-dimensional compact closed manifold with $\pi_2(M)$ satisfying the constraint of Section 2.2 (for the sake of concreteness one may take M to be S^3). The degree 3 form Θ_3 of Eq. (4.3), which gives a class in $H^3(\mathcal{A}/\mathcal{G})$, pulls back to $\mathcal{A}$. We then transgress to obtain the corresponding 2-cocycle on $L(\mathcal{G})$. In [6] we showed that the resulting cocycle was cohomologous to the Faddeev–Mickelsson cocycle [11,14,16] here denoted μ_2:

$$\mu_2(\epsilon_1,\epsilon_2) = -\frac{\mathrm{i}}{2\pi^3} \int_M str(A \wedge \mathrm{d}_A\epsilon_1 \wedge \mathrm{d}_A\epsilon_2) \tag{4.5}$$

where $\epsilon_i \in L(\mathcal{G})$.

Now by the construction of Section 2.2 Θ_3 defines the tautological bundle 1-gerbe on $\mathcal{A}/\mathcal{G}$:

$$Q_1 \longrightarrow Y^{[2]} \overset{\pi_i}{\rightrightarrows} Y^{[1]} = \mathcal{P}(\mathcal{A}/\mathcal{G}) \longrightarrow \mathcal{A}/\mathcal{G}. \tag{4.6}$$

Here $Q_1 \longrightarrow Y^{[2]}$ is a line bundle over the the loop space of $\mathcal{A}/\mathcal{G}$, $S^1(\mathcal{A}/\mathcal{G})$, with first Chern class given by

$$2\pi \mathrm{i} \int_{S^1} ev^*(\Theta_3) = \omega_2, \tag{4.7}$$

where $ev : S^1(\mathcal{A}/\mathcal{G}) \times S^1 \longrightarrow \mathcal{A}/\mathcal{G}$ is the evaluation map.

We may also pull back ω_2 to $\mathcal{A}$ (we denote the pull back form by the same symbol). It then defines a trivial bundle 1-gerbe on $\mathcal{A}$ which is a line bundle on $Y^{[2]} = S^1(\mathcal{A})$ given explicitly as follows. Take triples consisting of a loop $A \in S^1(\mathcal{A})$, a path γ joining a fixed base point to A and $z \in C$ (note that we need to fix a base point in each connected component). Consider equivalence classes:

$$[(A, \gamma, z)] \tag{4.8}$$

under the equivalence relation given by

$$(A_1, \gamma_1, z_1) \sim (A_2, \gamma_2, z_2) \tag{4.9}$$

if and only if $A_1 \equiv A_2$, and $z_1 = z_2 \exp(\int_\sigma \omega_2)$, where σ is a surface in $\mathcal{A}$ with boundary $\gamma_1 * \gamma_2^{-1}$. An easy calculation reveals that the gauge group $\mathcal{G}$ acts on this line bundle by

$$g.[(A, \gamma, u)] = \left[\left(g.A, g.\gamma, u \exp\left(\int_\gamma \alpha \right) \right) \right]. \tag{4.10}$$

Since $S^1(\mathcal{A})$ is also affine, we have

$$\omega_2(A) = \delta\rho(A), \tag{4.11}$$

where ρ is a one-form on $S^1(\mathcal{A})$. Moreover $\rho|_{S^1(\mathcal{G}).\mathcal{A}}$ defines the same cohomology class as

$$\int_{S^1} ev^*(\mu_2). \tag{4.12}$$

(One may think of the S^1 variable as introducing a periodic time dependence and hence considering time-dependant gauge potentials and gauge transformations). Of course on $S^1(\mathcal{A})$ the bundle 1-gerbe is trivial. So the corresponding 2-cocycle is cohomologically trivial. But the anomaly still manifests itself since the two-form ω_2 of Eq. (4.11) on $S^1(\mathcal{A})$ is not invariant under the action of $S^1(\mathcal{G})$. The corresponding anomaly can be obtained from (4.12).

Another interesting bundle 1-gerbe on $\mathcal{A}/\mathcal{G}$ is given by the following construction. Let γ be a closed path in M starting and ending at x_0. Then the holonomy along γ gives a map:

$$h : \mathcal{A}/\mathcal{G} \longrightarrow G. \tag{4.13}$$

Therefore we have $h^*(\omega_3) \in H^3(\mathcal{A}/\mathcal{G}, Z)$ where ω_3 is the form (2.10). Notice that (4.13) also gives the obvious map

$$\tilde{h} : S^1(\mathcal{A}/\mathcal{G}) \longrightarrow S^1(G). \tag{4.14}$$

As explained in Section 2, the extension of the loop group $S^1(G)$ defined by ω_3 is a bundle 1-gerbe $Q_2 \to S^1(G)$ on the compact Lie group G. Pulling back Q_2 on $S^1(G)$ to $S^1(\mathcal{A}/\mathcal{G})$ using $\tilde{h}$, Eq. (4.14), we obtain a bundle 1-gerbe on $\mathcal{A}/\mathcal{G}$. We believe that this bundle 1-gerbe is stably isomorphic to Q_1 (see [6] for the notion of stable isomorphism) but do not have a proof as yet.

5. 3-Cocycles and bundle 2-gerbes

In [10] Jackiw argued that a non-vanishing 3-cocycle in quantum field theories is a measure of non-associativity. There is a simple quantum mechanical example, arising from a point particle with charge e, at a point $\boldsymbol{r}$ moving in an external magnetic field $\boldsymbol{B}$, which is not necessarily divergence free. Geometrically speaking the Bianchi identity fails. Non-associativity is however a paradoxical interpretation in an operator algebra and the correct mathematical point of view is to realise that when the Jacobi identity breaks down in the sense that the Lie triple brackets give a 3-cocycle, one is dealing with an obstruction to the existence of a Lie algebra extension. Motivated by these quantum field theory examples, in [3,5], 3-cocycles as obstructions to the existence of an extension of one Lie algebra by another were derived by a C^*-algebra method. It was found that the underlying reason for the occurrence of this cocycle in chiral gauge theories is that the equal time formalism is too singular in 3 + 1-dimensions to permit the definition of a consistent Lie algebra of canonical fields. These anomalies do not as yet have a geometric interpretation but it is tempting to speculate that the notion of a bundle 2-gerbe may provide such an interpretation. To see how this might work we consider an example for a four-dimensional manifold M.

Henceforth, we suppose $M = S^4$, that P is a principal bundle over M, $\mathcal{A}$ is the irreducible connection space, $\mathcal{G}$ the gauge group. We constructed in Section 4.1 a degree 4 cohomology class $[\Theta_4]$ in $H^4(\mathcal{A}/\mathcal{G})$. The results of Section 3 show that Θ_4 defines the tautological bundle 2-gerbe on $\mathcal{A}/\mathcal{G}$. On the other hand the transgression of Θ_4, say μ_3, is a 3-cocycle on the Lie algebra of the gauge group. As such it may be thought of as an obstruction to the existence of a Lie algebra extension.

However using the tautological construction we can think of Θ_4 as giving rise to an extension of the loop group of $\mathcal{G}$, $S^1(\mathcal{G})$. To see this we fix a base point in each connected component of $\mathcal{A}/\mathcal{G}$ and $\mathcal{G}$. Next we consider the transgression:

$$H^3(\mathcal{G}, Map(\mathcal{A}, R)) \longrightarrow H^2(S^1(\mathcal{G}), Map(\mathcal{A}, R)). \tag{5.1}$$

Under this map the cocycle $\mu_3 \in H^3(\mathcal{G}, Map(\mathcal{A}, R))$ gives a 2-cocycle Λ on $S^1(\mathcal{G}$ with coefficients in $Map(\mathcal{A}, R)$. Now Λ defines an extension of $S^1(L(\mathcal{G}))$ by the abelian Lie algebra $Map(\mathcal{A}, R)$.

Proposition 5.1. *The 3-cocycle μ_3 gives rise to an extension of the Lie algebra of the loop group of the gauge group, $S^1(L(\mathcal{G}))$. The new Lie bracket on $S^1(L(\mathcal{G}))$ is given by the following formula:*

$$[(\epsilon_1, f_1), (\epsilon_2, f_2)] = ([\epsilon_1, \epsilon_2], \epsilon_1 \cdot f_2 - \epsilon_2 \cdot f_1 + \Lambda(\epsilon_1, \epsilon_2)), \tag{5.2}$$

where $[\epsilon_1, \epsilon_2]$ is the pointwise Lie bracket and $\epsilon_i \cdot f_j$ is the induced action of the Lie algebra of the gauge group on functions on $\mathcal{A}$.

It would be interesting to understand whether the 3-cocycle studied in [5] can be given a geometric understanding using the methods of this paper.

Remark. Interesting examples of de Rham forms on $\mathcal{A}/\mathcal{G}$ arise from the study of the descent equations [15] which are examples of a general approach to constructing secondary characteristic classes due to Chern and Simons. The Chern–Simons classes on $\mathcal{A}$ studied in [15] are closed and $\mathcal{G}$ invariant and so push down to forms on $\mathcal{A}/\mathcal{G}$. They are not, however, pull backs of closed forms on this quotient space. In the case of a three-dimensional base manifold M, the appropriate Chern–Simons three-form on $\mathcal{A}$ maybe written as $\mathrm{d}\rho$ where ρ defines, using the transgression procedure of Section 4, a 2-cocycle on the Lie algebra of the gauge group cohomologous to the Faddeev–Mickelsson 2-cocycle. This raises the question of whether the Chern–Simons forms of [15] pushed down to forms on $\mathcal{A}/\mathcal{G}$ are related in any way to the class $[\Theta_3]$ of the bundle 1-gerbe of Section 2. We have not been able to find such a relationship.

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